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Threshold Effects on the QCD Coupling ¹

$$\alpha_{\overline{MS}}$$

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Abstract:

The matching condition which determines the effect of a heavy quark threshold on the running of the QCD coupling $\alpha_{\overline{MS}}$ is reviewed. The matching scale is arbitrary to some extent. However, this affects the value of $\alpha_{\overline{MS}}$ away from the threshold region only marginally.

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1 Introduction

The QCD coupling at the scale of the Z resonance is presently obtained with an error of the order of about 5% (for reviews, see [1]). At this level of precision a careful discussion of errors, especially theoretical errors is necessary. When referring to α_s the QCD coupling defined in the \overline{MS} renormalization scheme is usually meant since most of the α_s determinations are done by comparing data with formulae obtained in this scheme. This note addresses a topic which is relevant in view of the precision one aims at and which is quite often incorrectly treated in the literature: namely, how to evolve $\alpha_{\overline{MS}}$ which was obtained, say, at the scale of the τ mass across heavy quark "thresholds" up to the scale of the Z mass (or vice versa and how to estimate the error on $\alpha_{\overline{MS}}$ associated with threshold crossing. For the correct treatment of this problem one has to use the matching relation for the \overline{MS} coupling which was obtained to order α_s^2 [2] and to order α_s^3 [3] quite some time ago. Below the use of this matching relation is illustrated with the example above.

2 Matching relations for $\alpha_{\overline{MS}}$ and for the light quark masses

As is well-known minimal subtraction renormalization (MS) provides, apart from some calculational advantages, a gauge- and vertex-independent, albeit physically unintuitive, definition of the QCD coupling. However, decoupling of heavy quarks in these schemes is not manifest. In the energy region $\mu^2 \ll m^2$, where m is the mass of a heavy quark, the contribution of this quark to an observable blows up like some power of $\ln(m/\mu)^2$ in a given order of perturbation theory. In the case of the top quark these logarithms can become quite large and signal a breakdown of perturbation theory. This behaviour is related to the fact that there is also no decoupling of a heavy quark flavour in the β function which governs the scale dependence of α_s . In order to establish decoupling in the MS schemes one has to resum these logarithms. In practice this is done by matching the full f flavour theory and the effective light flavour theory in the energy region below the heavy quark thresholds. Consider without loss of generality QCD with $f - 1$ light quarks and one heavy quark. In the region where the squared momenta of

a certain process are much smaller than m^2 the decoupling theorem tells us that we may calculate this process also in the "light", i.e., the $f - 1$ flavour theory up to terms of order $1/m$. By performing such calculations both in the full and in the light theory in the minimal subtraction scheme one can match both theories and thereby obtain relations between the parameters, i.e., the coupling and the light quark masses, of both theories. (For a review, see [5].) In the loop expansion with respect to the f flavour theory these relations have the structure [3, 4, 5]:

$$\alpha_{f-1}(\mu) = \alpha_{f-1}(\mu) \left[1 + \sum_{k=1}^{\infty} C_k(x) \left(\frac{\alpha_f(\mu)}{\pi} \right)^k \right], \quad (1)$$

$$m_{\ell}^{(f-1)}(\mu) = m_{\ell}^{(f)}(\mu) \left[1 + \sum_{k=1}^{\infty} H_k(x) \left(\frac{\alpha_f(\mu)}{\pi} \right)^k \right], \quad (2)$$

$$\ell = 1, \dots, f - 1$$

where α_{f-1} and $m_{\ell}^{(f-1)}(\mu)$, respectively α_f and $m_{\ell}^{(f)}(\mu)$ denote the QCD coupling and the running light quark masses in $f - 1$ flavour, respectively f flavour QCD for a specific MS-renormalization prescription. Furthermore μ denotes the renormalization scale,

$$x = \ln(m(\mu)/\mu)^2, \quad (3)$$

where $m(\mu)$ is the heavy quark mass defined in the f flavour theory, and $C_k(x)$ and $H_k(x)$ are polynomials in x of degree k . Note the following features of the matching relations (1), (2):

- a) The polynomials C_k and H_k are gauge-independent and independent of the light quark masses.
- b) The structure of (1) and (2) is dictated by the perturbative renormalization group; i.e., terms which behave like an (inverse) power in the heavy quark mass are absent. (In the relation between Green functions such terms are, of course, present.)
- c) The matching of the parameters of the $f - 1$ and f flavour theories is done at a scale $\mu = \mu^*$. This scale is arbitrary to some extent. As usual, μ^* should

be chosen such that the perturbation expansions can be kept under control.
d) For several heavy quark flavours eqs. (1) and (2) can be applied subsequently.

Eq. (1) was calculated to order α_f^3 in [3]. In the following we refer to the \overline{MS} scheme supplemented by the convention that the trace of the unit matrix (in spinor space) is kept equal to 4 in d space-time dimensions. (Note that there are additional terms in eq.(4) if one uses another convention; see [3].) Then to this order:

$$\alpha_{f-1}(\mu) = \alpha_f(\mu) \left[1 + \frac{x}{6} \frac{\alpha_f(\mu)}{\pi} + \left(\frac{x^2}{36} + \frac{11x}{24} + \frac{7}{72} \right) \left(\frac{\alpha_f(\mu)}{\pi} \right)^2 \right]. \quad (4)$$

The criterion for the matching scale μ^* , when computing α_{f-1} from α_f or vice versa with this formula, is that $|x|$ must not become very much larger than one. An often used choice is the mass of the heavy quark, $\mu^* = m$ ($\mu = m$). Then to order α_f^2 :

$$\alpha_{f-1}(m) = \alpha_f(m) \quad (5)$$

holds; whereas to order α_f^3 :

$$\alpha_{f-1}(m) = \alpha_f(m) + 7\alpha_f^3(m)/72\pi^2. \quad (6)$$

That is, the \overline{MS} coupling is *not* continuous ² at $\mu^* = m$. Of course one may compute from eq.(4) the matching scale where the higher order terms on the right hand side of (4) cancel; but for carrying out the matching procedure it is not necessary to obtain this value. Instead of $\mu^* = m$ we may use some other matching scale, for instance the threshold energy for quark-antiquark production, $\mu^* = 2m$. Then we get from (4):

$$\alpha_{f-1}(2m) = \alpha_f(2m) + \frac{\ln 4}{6} \frac{\alpha_f^2(2m)}{\pi} + \left[\frac{(\ln 4)^2}{36} + \frac{11\ln 4}{24} + \frac{7}{72} \right] \frac{\alpha_f^3(2m)}{\pi^2}, \quad (7)$$

This example illustrates that one gets in general a discontinuity already at next-to-leading order. Note that the "continuity requirement" of the \overline{MS}

²Eq. (6) yields also the relation between the QCD scales Λ_{f-1} and Λ_f in the \overline{MS} scheme [3, 6].

couplings α_{f-1} and α_f at the matching point μ^* which is often used in the literature is in general incorrect. Nevertheless, most of the applications to-date are based on next-to-leading order calculations. To this order eq.(5) holds which yields a continuous coupling at $\mu^* = m$. However, for estimating the error associated with the arbitrariness of the matching point – by varying μ^* in a some range around m – eq. (4) has to be used. The ”continuity requirement” overestimates this error.

In the \overline{MS} scheme as specified above the relation between the light quark masses in the $f - 1$ and f flavour theories is given to order α_f^2 by [5]:

$$m_\ell^{(f-1)}(\mu) = m_\ell^{(f)}(\mu) \left[1 + \frac{1}{12} \left(x^2 + \frac{5x}{3} + \frac{89}{36} \right) \left(\frac{\alpha_f(\mu)}{\pi} \right)^2 \right]. \quad (8)$$

Hence in next-to-leading order $m_\ell^{(f-1)}(\mu) = m_\ell^{(f)}(\mu)$ holds.

If one uses a mass-independent momentum subtraction scheme for the definition of the QCD coupling then there is also a non-trivial matching relation. This relation depends on the vertex used to define α_s [5, 7]. In these schemes the coupling is discontinuous at $\mu^* = m_{heavy}$ already at next-to-leading order.

3 Computing $\alpha_5(m_Z)$ from $\alpha_3(m_\tau)$

In order to illustrate the use of the matching relation (4) I shall now calculate the \overline{MS} coupling at the Z resonance, $\alpha_5(m_Z)$, from the recently determined coupling $\alpha_3(m_\tau)$ [8, 9, 10, 11] and determine the error associated with the arbitrariness of the threshold crossing points μ^* . This exercise was recently also made in [13]. The semihadronic to electronic τ decay ratio R_τ was computed in 3 flavour QCD [8, 10] (recall that the perturbative contributions are known to order α_s^3 [12]) and the value $\alpha_3(m_\tau) = 0.36 \pm 0.03$ was obtained in [8, 9]. (Alternatively one may compute R_τ also in \overline{MS} renormalized 4 flavour QCD with a massive c quark and thereby determine $\alpha_4(m_\tau)$.)

The calculation of $\alpha_5(m_Z)$ from this value can be done in many different ways. Both for $m = m_c$ and $m = m_b$ the coefficients of the higher order terms in eq.(4) remain quite small if we choose μ^* between these two mass values. Therefore we may choose only one matching point, say $\mu^* = m_\tau$ and

compute α_4 and α_5 by means of eq.(4) at this scale:

$$\alpha_3(\mu^* = m_\tau) \longrightarrow \alpha_4(\mu^* = m_\tau) \longrightarrow \alpha_5(\mu^* = m_\tau) \quad (9)$$

and then evolve $\alpha_5(m_\tau)$ with the 3-loop β function of 5 flavour QCD to $\alpha_5(\mu = m_Z)$. The solution of the 3-loop evolution equation can be represented as [6]:

$$\begin{aligned} \frac{1}{\alpha_f(\mu')} &= \frac{1}{\alpha_f(\mu)} - b_1 \ln(\mu'/\mu) - \frac{b_2}{b_1} \ln\left(\frac{\alpha_f(\mu')}{\alpha_f(\mu)}\right) \\ &\quad - \frac{1}{b_1^2} (b_3 b_1 - b_2^2) [\alpha_f(\mu') - \alpha_f(\mu)] + \mathcal{O}(\alpha_f^2) \end{aligned} \quad (10)$$

with

$$\begin{aligned} b_1 &= -\frac{1}{2\pi} \left(11 - \frac{2f}{3}\right) & b_2 &= -\frac{1}{4\pi^2} \left(51 - \frac{19f}{3}\right) \\ b_3 &= -\frac{1}{64\pi^3} \left(2857 - \frac{5033f}{9} + \frac{325f^2}{27}\right). \end{aligned} \quad (11)$$

Using $\alpha_5(m_\tau)$ as input in eq.(10) and iterating a few times one gets $\alpha_5(m_Z)$. Another possibility is to convert

$$\alpha_3(\mu^* = m_\tau) \longrightarrow \alpha_4(\mu^* = m_\tau), \quad (12)$$

evolve α_4 within 4 flavour QCD to $\mu^* = m_b$, then convert

$$\alpha_4(\mu^* = m_b) \longrightarrow \alpha_5(\mu^* = m_b), \quad (13)$$

and then scale α_5 to $\mu = m_Z$. We shall use this procedure. As further ingredient for eq.(4) one needs the \overline{MS} masses of the c and b quark at the scale μ^* . We use the values given in [14] which correspond to $m_c(m_\tau) = 1.3 \pm 0.1$ GeV and $m_b(m_b) = 4.3 \pm 0.1$ GeV. The values of $m_{c,b}$ at other scales can be obtained using the renormalization group.

In order to estimate the error associated with the arbitrariness of μ^* one may first keep $\mu^* = m_\tau$ fixed but vary the b "threshold" between, say, 2 GeV

$\leq \mu^* \leq 20$ GeV. Using the central value $\alpha_3(m_\tau) = 0.36$ as input one obtains $\alpha_5(\mu = m_Z) = 0.123$ and this value varies by less than 0.6% when varying μ^* in the above range. Keeping the second threshold value at $\mu^* = m_b$ and varying the c threshold between $1.7 \text{ GeV} \leq \mu^* \leq 4 \text{ GeV}$ one arrives at essentially the same result. This is in agreement with the conclusions of [13]. With $\alpha_3(m_\tau) = 0.36 \pm 0.03$ one then obtains $\alpha_5(m_Z) = 0.123 \pm 0.004 \pm 0.001$ (cf. also [9]) where the last error is a conservative estimate of the uncertainty associated with the matching points.

In summary, in the evolution of the \overline{MS} coupling α_s across heavy quark thresholds the matching condition eq.(4) comes into play. Although the matching scales are not fixed the resulting error on the coupling is very small as exemplified with the above example.

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